Preference-Based Nonlinear Normalization for Multiobjective Optimization

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Abstract. Normalization is commonly used in multiobjective evolutionary algorithms (MOEAs) in order to handle multiobjective optimization problems with differently-scaled objectives. The goal of normalization is to obtain uniformly-distributed solutions over the entire Pareto front. However, in practice, such a uniform solution set may not be a well-distributed solution set for decision making when the desired distribution of solutions is not uniform. To obtain a well-distributed solution set that meets the desired distribution, in this paper, we propose a preference-based nonlinear normalization method that transforms the objective space based on the probability integral transform theorem. As a result, the use of a standard MOEA to search for uniformly-distributed solutions in the transformed objective space leads to a desired well-distributed solution set. The proposed method is incorporated in three different MOEAs (i.e., a Pareto dominance-based MOEA, a decomposition-based MOEA, and an indicator-based MOEA). Experimental results demonstrate the flexibility and effectiveness of the proposed method. Our code is available at https://github.com/linjunhe/moea-pn.

Keywords: Evolutionary multiobjective optimization (EMO) · Preference incorporation · Decision making · Normalization.

1 Introduction

Real-world multiobjective optimization problems (MOPs) usually have multiple conflicting and differently-scaled objectives [20, 29]. To solve such problems, various multiobjective evolutionary algorithms (MOEAs) have been proposed in recent years [21]. In recently proposed MOEAs, normalization is usually used before environmental selection to handle badly-scaled MOPs [6, 15, 23, 24, 38]. Various studies have been conducted to examine and improve normalization methods (see Section II-B). In such studies on normalization, researchers usually implicitly assume that the desired distribution of solutions on each objective is uniform. As a result, the goal of normalization is to obtain uniformly-distributed solutions over the entire Pareto front in a normalized objective space.

However, for some real-world applications where the law of diminishing returns holds, a uniformly-distributed solution set may not be a well-distributed solution set for decision making [10]. For example, let us assume that we are looking for a car for our personal use based on the following two objectives: maximization of the maximum speed and minimization of the price. For the first objective, presentation of uniformly-distributed solutions to the decision maker may be acceptable when he/she does not articulate any preferences. However, for the second objective, the price distribution of available cars is generally not uniform but positively skewed [10, 34] as illustrated in Fig. 1(a). This naturally raises a question: which is a better solution set between the following two sets of prices ($\times$1000$) of 10 candidate cars for decision making?

- Positively skewed distribution (see Fig. 1(b)): $A = \{40, 70, 90, 100, 120, 140, 170, 210, 270, 400\}$.
- Uniform distribution (see Fig. 1(c)): $B = \{40, 80, 120, 160, 200, 240, 280, 320, 360, 400\}$.

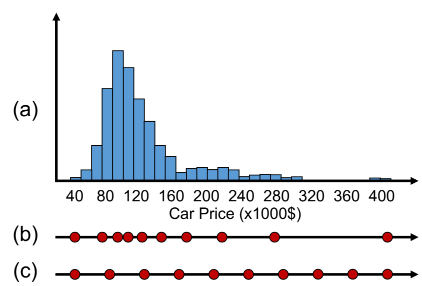


Fig. 1. Illustration of (a) histogram of car price, (b) positively skewed solutions (well-distributed for decision making), and (c) uniformly-distributed solutions.

As pointed out in [34], the presentation of candidate car set A with a biased distribution will be more useful for most people than candidate car set B with a uniform distribution. This is because the distribution of A is similar to the distribution of cars in the car market.

To obtain a well-distributed solution set like A for decision making, in this paper, a preference-based nonlinear normalization method is proposed. The contributions of this paper can be summarized as follows.

- We propose a preference-based nonlinear normalization method. Based on the preference (i.e., the desired distribution of solutions based on collected data), the objective space is transformed according to the probability integral transform theorem, such that the search of uniformly-distributed solutions in the transformed space results in well-distributed solutions in the original objective space for decision making.

- We discuss the relation between the proposed normalization method and the conventional linear normalization method. Experimental results show that the conventional linear normalization method is a special case of the proposed method.
- The proposed method can be incorporated in any existing MOEAs in a plug-in manner. This is different from existing preference incorporation methods (see Section II-C) that need a specific modification in the environmental selection mechanism of each MOEA. We incorporate the proposed method in different MOEAs to demonstrate its flexibility and effectiveness.

The rest of the paper is organized as follows. Preliminary knowledge on multiobjective optimization, linear normalization, and preference-based MOEAs are presented in Section 2. In Section 3, the proposed preference-based nonlinear normalization is presented, and its relation to linear normalization is discussed. In Section 4, comprehensive experiments are conducted to verify the discussed relation and to demonstrate the flexibility and effectiveness of the proposed method. In Section 5, we conclude the paper.

2 Preliminaries

2.1 Multiobjective Optimization Problem

A multiobjective optimization problem (MOP), which aims to minimize m conflicting objectives at the same time, can be written as follows.

$$\begin{aligned} \text{Minimize} \quad & \mathbf{f}(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_m(\mathbf{x}))^T, \\ \text{subject to} \quad & \mathbf{x} \in \Omega, \end{aligned} \tag{1}$$

where $f_i(\mathbf{x})$ is the i-th objective function and $\mathbf{x}$ is an n-dimensional decision vector in the feasible region $\Omega \subseteq \mathbb{R}^n$. Due to the conflicting nature of the objectives, the MOP has a set of Pareto optimal solutions, called the Pareto set. The image of the Pareto set in the objective space is called the Pareto front (PF).

2.2 Linear Normalization

To deal with MOPs with differently-scaled objectives, objective space normalization is usually performed before environmental selection of an MOEA. Each objective function in (1) is usually linearly transformed as follows.

$$\widetilde{f}_i(\mathbf{x}) = \frac{f_i(\mathbf{x}) - z_i^{\text{lb}}}{z_i^{\text{ub}} - z_i^{\text{lb}}}, i \in \{1, 2, \ldots, m\}, \tag{2}$$

where $\widetilde{f}_i(\mathbf{x})$ is the i-th normalized objective function, and z_i^{lb} and z_i^{ub} are the lower and upper bounds of the i-th objective function, respectively.

Investigation on normalization has attracted a lot of researchers' attention. As pointed out in [16, 18], normalization methods can affect the performance of decomposition-based MOEAs in both positive and negative ways. Fukumoto and Oyama [12] and Liu et al. [25] investigated the impact of normalization methods for constrained decomposition-based MOEAs and multi-modal MOEAs, respectively. He et al. [17] analyzed the relation between normalization methods and weight vector scaling methods for decomposition-based MOEAs. A metric was proposed in [13] for investigating normalization methods. To make use of the advantages of normalization and reduce its negative effects, several new normalization methods were proposed. Blank et al. [3] proposed a normalization method characterized by extreme point preservation. Dynamic normalization methods were designed based on a sigmoid function in [14] or a step function in [28]. Wang et al. [36] proposed to use surrogate-based search to improve normalization bounds. Among these studies on normalization, researchers usually implicitly assume that the desired distribution of solutions on each objective is uniform. The proposed preference-based nonlinear normalization method in this paper does not rely on this assumption.

2.3 Preference Incorporation

Generally, decision makers are often interested in a small region of the PF instead of the entire PF, known as the region of interest (ROI). To search for the ROI, various approaches have been proposed to incorporate preference into MOEAs. These approaches can be roughly divided into the following four categories.

- *Objective comparison-based approaches.* Relative importance of each objective can be described by weights specified by the decision maker, by linguistic labels obtained from pairwise comparisons between objectives, or by pairwise trade-off information provided by the decision maker. This information is then used to modify the Pareto dominance [11, 5], crowding operator [27], or quality indicator [40] to bias the population towards the ROI.
- *Solution ranking-based approaches.* Pairwise comparisons between solutions are made by the decision maker to learn a utility function. The learned utility function is then used to modify the dominance relation [7, 19], crowding operator [1], or both of them [4] in order to identity the ROI.
- *Reference point-based approaches.* The decision maker's preference is articulated by a reference point or a set of reference points. Solutions close to the reference point(s) are then prioritized by modifying the crowding operator [8, 26], dominance [39], or quality indicator [30] to guide the search towards the reference point(s).
- *Desirability function-based approaches.* For each objective, two thresholds (i.e., an absolutely satisfying objective value and a marginally infeasible objective value) are provided by the decision maker. These thresholds serve as parameters of desirability functions, by which the objective functions are transformed [35].

Most existing approaches directly incorporate preference information into the environmental selection mechanisms of MOEAs (e.g., modifying the dominance relation, the crowding operator, and the quality indicator). The proposed method focuses on the normalization part of MOEAs. The preference is incorporated by nonlinearly normalizing the objective space without any modifications on the environmental selection mechanisms of the original MOEAs. Note that the desirability function-based approach [35] also transforms the objective space. However, our method is different from [35] as follows.

1. The transformation in [35] is based on a desirability function and the decision maker is asked to provide an absolutely satisfying objective value and a marginally infeasible objective value. Our method transforms the objective space based on the probability integral transform theorem when the desired distribution of solutions (i.e., the distribution of collected data) is available.
2. The goal of [35] is to search for uniformly-distributed solutions in the ROI. Our method targets for a well-distributed solution set that meets the desired distribution of solutions for decision making.
3. The approach in [35] is designed for hypervolume-based MOEAs while our method can be integrated with any existing MOEAs.

3 Proposed Preference-Based Nonlinear Normalization

In this section, the proposed preference-based nonlinear normalization method is presented. The goal of the proposed method is to adjust the distribution of solutions for each objective. For each objective, the desired distribution of solutions can be either inferred from collected data or specified by the decision maker.

With the desired distribution, the objective is transformed by the corresponding cumulative distribution function (CDF). This transformation can be understood by the probability integral transform theorem [33]: Suppose that a random variable X has a continuous distribution for which the CDF is Φ. Then $\Phi(X)$ is a random variable having a standard uniform distribution. This theorem ensures that the desired distribution of solutions for the original objective function is converted into a uniform distribution after such transformation. As a result, we can use a standard MOEA to search for uniformly-distributed solutions in the transformed objective space. The obtained solutions are well-distributed in the original objective space. The details of the proposed nonlinear transformation are presented as follows.

Collected data. The desired distribution of solutions can be modeled by collected data like Fig. 1(a). Since the original distribution of collected data is usually unknown, we cannot compute the exact CDF. Instead, we compute the empirical CDF. The empirical CDF is an estimate of the CDF that generates the points in the sample, and it converges with the probability of one to the original distribution according to the Glivenko–Cantelli theorem [32]. For a data set $\{x_1, x_2, ..., x_n\}$, the empirical CDF is calculated as follows:

$$\Phi(x) = \frac{1}{n} \left| \{x_i | x_i \leq x, i = 1, 2, \ldots, n\} \right|, \tag{3}$$

where $|\cdot|$ measures the cardinality of a set. In other words, the value of the empirical CDF at a given point x is the proportion of observations that are less than or equal to x.

Note that the empirical CDF is a step function that makes a discrete jump of size $1/n$ at each of the n data points. Due to its discreteness, the empirical CDF cannot be directly used as a continuous transformation function for each objective. To transform objective values at points other than the original data points, linear interpolation is performed by connecting each midpoint of adjacent two jumps (e.g., adjacent data points) in the empirical CDF to smooth the step function.

Preference distribution. When data are unavailable, the distribution can be specified by the decision maker. We use the beta distribution to model the decision maker's preference due to its ability to take a great diversity of shapes using only two positive real number parameters α and β. By specifying the two parameters, the decision maker can express his/her preference for the desired distribution of solutions for each objective as shown in Fig. 2(a). For example, the distribution with $\alpha = 1$ and $\beta = 10$ means that the decision maker prefers to have more solutions with small objective values. As an extreme case, $\alpha = \beta = 1$ means that the decision maker has no preference about the distribution of desired solutions.

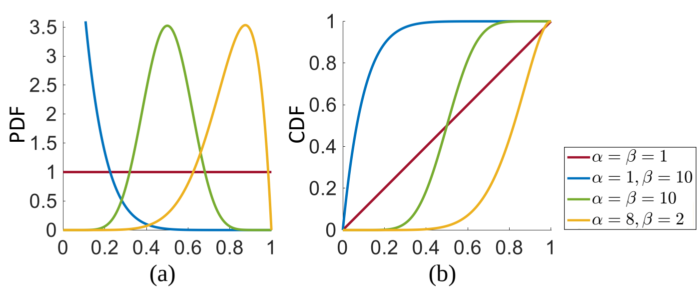


Fig. 2. Example of (a) probability density functions (PDFs) and (b) their corresponding cumulative distribution functions (CDFs) of the beta distribution with different values of α and β.

With the articulated preference distribution as a beta distribution, the objective function is transformed by the following transformation function:

$$\Phi(x \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt, \tag{4}$$

where $B(\cdot)$ is the beta function. In practice, Eq. (4) is the cumulative distribution function (CDF) of the beta distribution. Fig. 2(b) shows the corresponding CDFs for the PDFs in Fig. 2(a). The slope in each CDF shows how quickly the objective value is changing after transformation. A steeper slope in the CDF means that more solutions are preferred while a gentler slope means that less solutions are preferred.

If the decision maker has no preference for an objective, the desired distribution is specified as a uniform distribution (i.e., $\alpha = \beta = 1$). For the uniform distribution $U(a, b)$, where a and b are the minimum and maximum values, its CDF is $\Phi(x) = (x - a)/(b - a)$ for $a \leq x \leq b$. By replacing x with $f_i(\mathbf{x})$, we have $\Phi(f_i(\mathbf{x})) = (f_i(\mathbf{x}) - a)/(b - a)$. When a and b are the lower and upper bounds as in (2), such transformation is exactly the same as the linear normalization. As shown in Fig. 2(b), the CDF of the uniform distribution (i.e., $\alpha = \beta = 1$) is linear. The proposed transformation performs a linear mapping from the original objective values to the range $[0, 1]$, which is exactly the same as the linear normalization. That is, when the uniform distribution is specified, the proposed normalization method is equivalent to the common linear normalization method.

Incorporation in MOEAs and indicators. The proposed nonlinear normalization method can be easily incorporated into any MOEAs in a plug-in manner. This is because the proposed method focuses on the normalization part of MOEAs, which is an independent algorithmic component. In most existing preference-based MOEAs, the environmental selection mechanism of each algorithm is modified from its base MOEA. Such modification only works for that specific MOEA. On the contrary, the proposed method enables any MOEAs to search in a transformed objective space. In this paper, we incorporate the proposed normalization method into three MOEAs, one from each categories: SPEA2 [42] (a Pareto dominance-based MOEA), NSGA-III [6] (a decomposition-based MOEA), and SMS-EMOA [2] (an indicator-based MOEA). The resulting algorithms are denoted as SPEA2-PN, NSGA-III-PN, and SMS-EMOA-PN, respectively.

To evaluate the solutions obtained by preference-based MOEAs using reference points, Li et al. [22] transforms the obtained solutions using reference points, and the standard performance indicators are used. Inspired by [22], we use the proposed normalization method to transform the obtained solutions. After the transformation, the standard performance indicators can be used directly to evaluate the obtained solutions. In this paper, we use the hypervolume (HV) [43] and pure diversity (PD) [37] indicators. In the transformed objective space, these indicators are referred to as P-HV and P-PD.

4 Experimental Studies

In this section, we experimentally examine the proposed normalization method. First, the relation between the proposed method and the conventional linear normalization method is examined. Then, the proposed method is incorporated into different MOEAs and is examined on test problems under different preferences. We also visually examine the obtained solutions in the original and transformed objective spaces. Our experiments are conducted on PlatEMO [31]. In all the

Table 1. Average P-HV values over 51 runs obtained by the original SPEA2 and its two variants with different normalization methods.

Problem	SPEA2	SPEA2-N	SPEA2-PN
SZDT1	6.9959e-1 (2.39e-3) −	7.0343e-1 (1.04e-3) ≈	**7.0383e-1 (3.06e-4)**
SZDT2	4.2694e-1 (8.65e-4) −	**4.2935e-1 (2.65e-4)** ≈	4.2923e-1 (4.06e-4)
SZDT3	5.5570e-1 (4.68e-2) −	**5.7666e-1 (2.25e-2)** ≈	5.7659e-1 (3.01e-2)
+/ − / ≈	0/3/0	0/0/3	

examined algorithms, the population size is set to 20 in order to clearly show the effect of preference incorporation. The evaluation of $50,000$ solutions is used as the termination condition. Each algorithm is executed 51 times on each test problem. The Wilcoxon rank-sum test with a significance level of 0.05 is used to validate the statistical significance. The three symbols "+", "−", and "≈" mean that an algorithm is significantly better than, significantly worse than, or statistically similar to the baseline algorithm, respectively.

4.1 Relation to Linear Normalization

We have discussed the relation between the conventional linear normalization method and the proposed preference-based normalization method in Section 3. To experimentally demonstrate such relation, we compare the original SPEA2, SPEA2 with the linear normalization (denoted as SPEA2-N), and SPEA2 with the proposed preference-based normalization (SPEA2-PN). In SPEA2-PN, a uniform distribution (i.e., $\alpha = \beta = 1$) is applied to each objective. Since the proposed method does not modify the original SPEA2 and introduces no additional parameters, the algorithm parameters recommended in the original SPEA2 are used.

We choose ZDT1-3 [41] to examine the three algorithms. ZDT1 and ZDT2 have connected convex and connected concave PFs, respectively. ZDT3 has a disconnected PF with both concave and convex parts. Since our focus in this paper is badly-scaled MOPs, we modified the objectives of each test problem such that the first objective has the range $[0, 1000]$ and the second objective has the range $[0, 1]$. The modified MOPs with badly-scaled PFs are called SZDT1-3.

The average P-HV values and the standard deviation values are presented in Table 1. The original SPEA2 is significantly worse than SPEA2-PN (i.e., with the proposed normalization method) on the three badly-scaled test problems, while the results obtained by SPEA2-PN are statistically similar to these obtained by SPEA2-N (i.e., with the linear normalization method).

In Fig. 3, we show the the final population obtained by each algorithm on SZDT1 in a single run with the medium P-HV value. We can see that the original SPEA2 is not able to obtain uniformly distributed solutions on the PF of SZDT1, while SPEA2 with each normalization method obtains a uniform solution set. This is because the original SPEA2 maintains the diversity only relying the objective f_1 with a large scale since the value of the other objective f_2 is

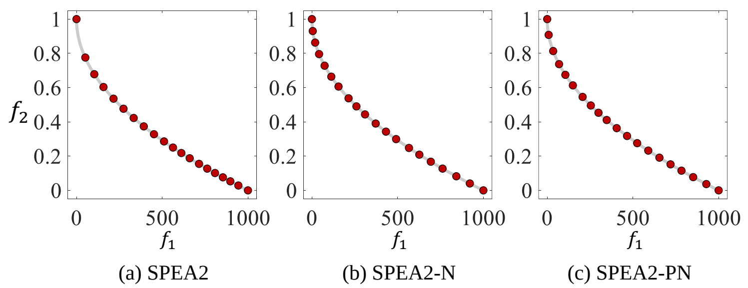


Fig. 3. Solutions obtained by the original SPEA2 and its two variants with different normalization methods on SZDT1.

neglectable due to the lack of normalization. Theoretically, the same results will be obtained from SPAE2-N and SPEA2-PN with $\alpha = \beta = 1$. Minor differences between Fig. 3(a) and Fig. 3(b) are due to randomness (e.g., different initial populations). These results clearly show that the proposed preference-based normalization method performs similarly as the conventional linear normalization method when the uniform distribution is applied for each objective in the proposed method.

4.2 Incorporation into Different MOEAs

In order to show the flexibility and effectiveness of the proposed normalization method, we incorporate it into SPEA2, NSGA-III, and SMS-EMOA. The resulting algorithms are denoted as SPEA2-PN, NSGA-III-PN, and SMS-EMOA-PN, respectively. We consider three specifications of preference (α, β) (see Fig. 2):

- Pref 1: $(1, 10)$ for f_1 and $(1, 1)$ for f_2,
- Pref 2: $(10, 10)$ for f_1 and $(1, 1)$ for f_2,
- Pref 3: $(10, 1)$ for f_1 and car price data [9] (see Fig. 1(a)) for f_2.

The MOEAs are examined by comparing their original version with its variant using the proposed normalization method. We use P-HV and P-PD to evaluate the ability of each algorithm to obtain solutions with the desired distribution. The results are presented in Table 2 and Table 3. Compared with the baseline algorithms, we can see that MOEAs with the proposed method is able to find better solutions under different preferences in terms of both P-HV and P-PD. That is, the proposed normalization method is able to change the search behavior of different MOEAs and enables them to search for solutions with the desired distribution and good convergence.

The obtained solutions in a single run with the median P-HV values among 51 runs of SMS-EMOA-PN under each preference setting are shown in Fig. 4 for SZDT1-3. We can see that solutions with different distributions are found when different preference settings are used regardless of the PF shape. For example, with the first type of preference, the obtained solutions concentrate on the upper left corner of the PF as shown in Figs. 4.

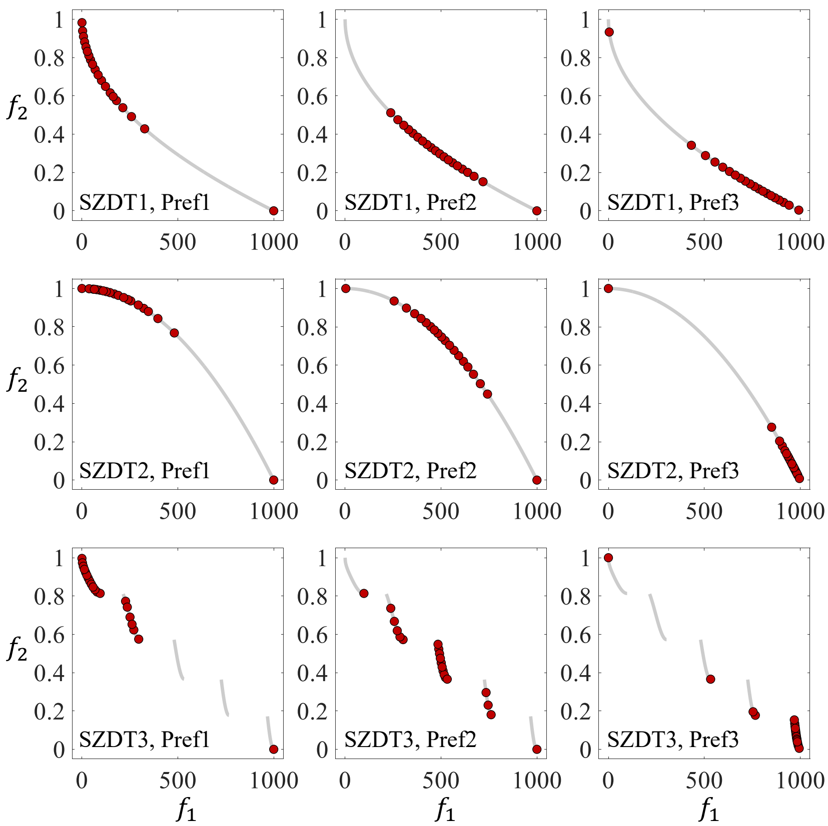


Fig. 4. Solutions obtained by SMS-EMOA-PN on SZDT1-3 with different preferences.

Table 2. Average P-HV values over 51 runs obtained by SPEA2, NSGA-III, SMS-EMOA and their variants incorporated with the proposed method.

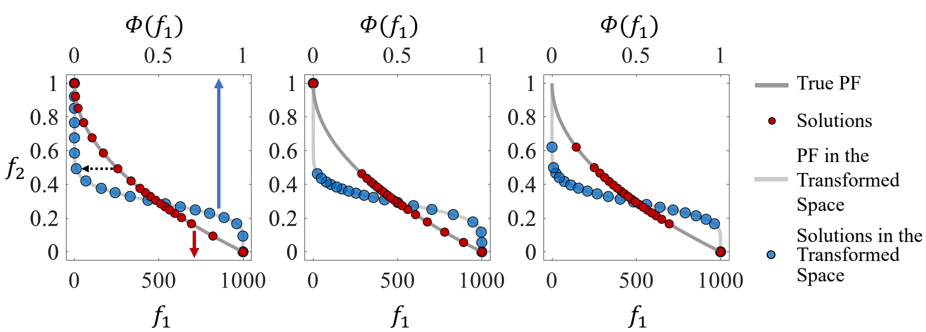
		SPEA2		NSGA-III		SMS-EMOA	
	Problem	Original	Proposed	Original	Proposed	Original	Proposed
Pref 1	SZDT1	3.6668e-1 −	**3.8084e-1**	3.4133e-1 −	**3.7897e-1**	3.5272e-1 −	**3.8395e-1**
	SZDT2	1.8173e-1 −	**1.8256e-1**	1.8217e-1 −	**1.8255e-1**	1.8035e-1 −	**1.8438e-1**
	SZDT3	2.7447e-1 −	**2.7805e-1**	2.6290e-1 −	**2.7609e-1**	2.7270e-1 ≈	**2.7527e-1**
Pref 2	SZDT1	7.3480e-1 −	**7.4153e-1**	7.2809e-1 −	**7.3734e-1**	7.3669e-1 −	**7.4386e-1**
	SZDT2	3.6777e-1 −	**3.7531e-1**	3.6882e-1 −	**3.7541e-1**	3.7044e-1 −	**3.8079e-1**
	SZDT3	5.9604e-1 ≈	**5.9973e-1**	5.9299e-1 −	**5.9639e-1**	5.9323e-1 −	**6.0059e-1**
Pref 3	SZDT1	7.6441e-1 −	**8.2860e-1**	7.0662e-1 −	**8.1430e-1**	7.6629e-1 −	**8.2969e-1**
	SZDT2	3.8772e-1 −	**4.8247e-1**	4.1563e-1 −	**4.7781e-1**	3.8747e-1 −	**4.8571e-1**
	SZDT3	2.5070e-1 −	**3.5554e-1**	2.3818e-1 −	**3.2576e-1**	1.8924e-1 −	**3.1968e-1**
	+/ − / ≈	0/8/1		0/9/0		0/8/1	

Table 3. Average P-PD values over 51 runs obtained by SPEA2, NSGA-III, SMS-EMOA and their variants incorporated with the proposed method.

		SPEA2		NSGA-III		SMS-EMOA	
	Problem	Original	Proposed	Original	Proposed	Original	Proposed
Pref 1	SZDT1	1.0294e+3 −	**1.2592e+3**	0.8314e+3 −	**1.2403e+3**	0.7924e+3 −	**1.1645e+3**
	SZDT2	3.7400e+2 −	**6.2384e+2**	3.6494e+2 −	**6.0748e+2**	3.5841e+2 −	**4.5318e+2**
	SZDT3	7.1915e+2 −	**8.3424e+2**	7.2647e+2 −	**9.7171e+2**	6.8284e+2 −	**8.8513e+2**
Pref 2	SZDT1	1.0118e+3 −	**1.2125e+3**	0.9623e+3 −	**1.0573e+3**	0.9885e+3 −	**0.9909e+3**
	SZDT2	1.2875e+3 −	**1.2915e+3**	1.3405e+3 −	**1.4146e+3**	1.0512e+3 −	**1.1568e+3**
	SZDT3	6.6243e+2 ≈	**6.7921e+2**	6.6102e+2 −	**7.6149e+2**	6.2770e+2 −	**7.3463e+2**
Pref 3	SZDT1	1.0248e+3 −	**1.2914e+3**	1.0318e+3 −	**1.2765e+3**	0.8897e+3 −	**1.0341e+3**
	SZDT2	1.0297e+3 −	**1.5667e+3**	1.0449e+3 −	**1.3278e+3**	0.9596e+3 −	**1.2921e+3**
	SZDT3	3.7906e+2 −	**8.3445e+2**	4.6532e+2 −	**8.1875e+2**	3.1576e+2 −	**8.5598e+2**
	+/ − / ≈	0/8/1		0/9/0		0/9/0	

4.3 Analysis in the Transformed Objective Space

In the previous subsection, we demonstrated the effectiveness and flexibility of the proposed normalization method. Here, we analyze the search behavior in the transformed objective space. The solution set obtained by each of the three algorithms (SPEA2-PN, NSGA-III-PN and SMS-EMOA-PN) on SZDT1 with Pref 2 is shown in each figure in Fig. 5 in the original and transformed spaces.



(a) SPEA2-PN (b) NSGA-III-PN (c) SMS-EMOA-PN

Fig. 5. Solutions obtained by (a) SPEA2-PN, (b) NSGA-III-PN, and (c) SMS-EMOA-PN on SZDT1 with Pref 2 in the original and transformed spaces.

To understand the search behavior of the proposed algorithms, we can take a look at the transformed objective space (with the top x-axis labeled as $\Phi(f_1)$) in Fig. 5. The true PF of SZDT1 (dark gray curve) is transformed to the light gray curve by the proposed nonlinear normalization method. Since the linear normalization is used for f_2, the f_2 value of each solution has no change whereas the location of each solution (i.e., f_1 value) is changed by the nonlinear transformation (see the dark dotted arrow in Fig. 5(a)). Uniformly-distributed blue solutions are obtained by SPEA2 in the transformed objective space. That is, by searching for the uniformly distributed blue solutions using SPEA2 with no modification in the transformed objective space, we can obtain the red solutions with desired distribution in the original space.

In addition, we can see that the distributions of the blue solutions by different algorithms are slightly different in Fig. 5. For SPEA2-PN, the obtained blue solutions are uniformly distributed due to the k-th nearest distance used in SPEA2. For NSGA-III-PN, the blue solutions close to each weight vector in NSGA-III are obtained. For SMS-EMOA-PN, the blue solutions that maximizes the HV value are obtained. This explains why different solution sets are obtained by the three algorithms in the original objective space even with the same preference.

5 Conclusion

In this paper, we proposed a preference-based nonlinear normalization method. Different from existing preference incorporation methods where the preference is incorporated by modifying the environmental selection mechanisms of existing MOEAs, we related preference with normalization. The preference is articulated in the form of a desired distribution of solutions, and then is incorporated into the proposed normalization method to transform the objective space according to the probability integral transform theorem. The proposed method enables any MOEAs to search for uniformly-distributed solutions in a transformed objective space and results in solutions with the desired distribution in the original objective space. We discussed the relation between the proposed normalization method and the conventional linear normalization method. We showed that when a uniform distribution is applied to each objective, the proposed method is the same as the linear normalization method. To show the flexibility of the proposed normalization method, we incorporated it into three MOEAs: SPEA2, NSGA-III, and SMS-EMOA. Experimental results showed that the standard MOEAs can find solutions of interest after incorporating the proposed method. We also analyzed the obtained solutions in the transformed space to clearly explain why the proposed method is effective. In this preliminary work, we only reported the results on two-objective MOPs. It is an interesting future research direction to examine the proposed method on MOPs with more than two objectives.

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